Chapter 16: Line Integrals

Section 1:

Definition 1:
A curve in space is a function $r(t)$ from a closed interval $[a,b]$ to space by: $r(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + l(t)\mathbf{k}$, $a \le t \le b$.
We shall always assume that $r'(t) = g'(t)\mathbf{i} + h'(t)\mathbf{j} + l'(t)\mathbf{k}$ exist and is continuous everywhere in $[a,b]$.

Note:
The curve of the line segment from *A* to *B* where $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ is

$$r(t) = [(1-t)a_1 + tb_1]\mathbf{i} + [(1-t)a_2 + tb_2]\mathbf{j} + [(1-t)a_3 + tb_3]\mathbf{k}$$

Definition 2:
Suppose $C:\ r(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + l(t)\mathbf{k}$ with $a \le t \le b$.
Given $f(x, y, z)$ a function of 3 variables which is continuous on *C*. The line integral of *f* over *C* is defined as

$\int_C f(x, y, z)dS = \lim_{\Delta S \to 0} \sum_{m=1}^{n} f(x_m, y_m, z_m)\Delta S_m$ where $(x_0, y_0, z_0), \ldots\ldots\ldots, (x_m, y_m, z_m)$ are successive points on *C*, ΔS_m is the length of *C* between $(x_{m-1}, y_{m-1}, z_{m-1})$ and (x_m, y_m, z_m), and $\Delta S = \max(\Delta S_1, \ldots., \Delta S_n)$.

Formula for evaluating line integral:

Suppose *C* and *f* are as above: Then: $\int_C f(x, y, z)dS = \int_a^b f(g(t), h(t), l(t))|v(t)|dt$

Where: $v(t) = r'(t)$.

Note:
$C_1 : r_1(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + l(t)\mathbf{k}$ is a curve in space. ($a \le t \le b$).
C_2 is a curve in space with the same path (as C_1) but in opposite direction. Then $r_2(t) = r_1(a + b - t)$

Section 2:

Definition 1:0
A vector field on an open region *D* in space is a function $F(x, y, z)$ from *D* to space given by $F(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$.
We shall always assume that *M*, *N*, and *P* have continuous first order partial derivatives.

Note:
If $f(x, y, z)$ is a function of three variables with continuous first and second order partial derivatives, then $\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$ is a vector field on the interior of Domain *f*.

Definition 2:
Suppose *D* is an open region in space.
Suppose $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field on *D*.
Suppose $C : r(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + l(t)\mathbf{k}$ ($a \le t \le b$) is a curve in *D*.

The unit tangent vector to C is defined as $T = \frac{v(t)}{|v(t)|}$ provided $v(t) \neq 0$.

The line integral of F over C is defined as $\int_C F.TdS$.

If F represents a force, then above integral is called the work integral. In this case, we write work = work done by F over C = $\int_C F.TdS$.

If F represents a velocity field, then above integral is called a flow integral. In this case, we write flow = flow of F along C = $\int_C F.TdS$.

If F represents a velocity field and C is closed $(\text{i.e. } r(a) = r(b))$, then above integral is called a circulation integral. In this case, we write circulation = circulation of F around C = $\int_C F.TdS$.

Formula for evaluating line integrals of vector field.

Suppose F and C are as above. Then: $\int_C F.TdS = \int_a^b F.\frac{dr}{dt}dt$

Notation:

Suppose $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field on some open region D in space.

Suppose $C : r(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + l(t)\mathbf{k}$ with $a \leq t \leq b$ is a curve in D, then $\int_C F.TdS = \int_C Mdx + Ndy + Pdz$

Section 3:

Definition 1:

Suppose $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field on an open region D in space.

$f(x, y, z)$ is a function of three variables which is differentiable everywhere in D.

If $\nabla f(x, y, z) = F(x, y, z)$ for all points (x, y, z) in D, we say that f is a potential function for F on D.

Definition 2:

Suppose F is a vector field on an open region D in space. We say that F is conservative if the line integral of F is path independent in D. i.e., if A and B are two points in D and C_1 and C_2 are two curves in D joining A to B, then $\int_{C_1} F.TdS = \int_{C_2} F.TdS$.

Theorem 1: Fundamental theorem of line integral (FTLI):

Suppose $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field on an open and connected region in space.

Then: F is conservative on D $\Leftrightarrow$ F has a potential function F on D.

In this case $\int_C F.TdS = f(B) - f(A)$ for every curve C in D joining A to B.

Definition 3:

A region D in space is connected if any two points in D can be joined by a curve which lies entirely in D.

Definition 4:

A region D in space is simply connected if any closed curve in D can be contracted to a point without ever leaving D.

Theorem 2: The Curl Test:

Suppose $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field on an open connected, simply connected region *D* in space.

Then: *F* is conservative on *D* $\Leftrightarrow$ Curl $F = 0$

Where $\text{Curl } F = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$

Definition 5:

Suppose *D* is an open region in space.

A differential form on *D* is an expression of the form $M(x, y, z)dx + N(x, y, z)dy + P(x, y, z)dz$.

We shall always assume that *M*, *N*, and *P* have continuous first order partial derivatives.

Definition 6:

$f(x, y, z)$ is a function of three variables with continuous first and second order partial derivatives. Then the differential of *f* is the differential form $f_x dx + f_y dy + f_z dz$.

Definition 7:

Suppose *D* is an open and connected region in space.

Suppose $Mdx + Ndy + Pdz$ is a differential form on *D*.

Then: If there is a function $f(x, y, z)$ such that $df = Mdx + Ndy + Pdz$ we say that the given differential form is exact $\Leftrightarrow$ $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative.

Section 4:

Definition 1:

$F(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is a vector field on an open region *R* in the plan.

Then, the divergence of *F* is defined as: $\text{div } F = \nabla . F = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} \right) . (M\mathbf{i} + N\mathbf{j}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

Note:

If $F(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, then $\text{Curl } F = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = -\frac{\partial N}{\partial z}\mathbf{i} + \frac{\partial M}{\partial z}\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\mathbf{k}$

So the **k** component of Curl *F* is $(\text{Curl } F).\mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$.

Theorem 1:

a. **Green's Theorem: Flux Divergence form:**

Suppose $F = M\mathbf{i} + N\mathbf{j}$ is a vector field in the plane.

Suppose *C* is a simple closed curve in the plane .Simple means, the position vector of the curve is one to one on (a, b).

Then: flux of *F* across $C = \iint_R \text{div } F dA$.

This can be written as $\oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy$

b. Green's Theorem: Circulation Curl form:

Suppose *F*, *C* and *R* are as above. Then:

Counterclockwise circulation of *F* around $C = \iint_R \left((\text{Curl } F).\mathbf{k} \right) dA$

This can be written as: $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$